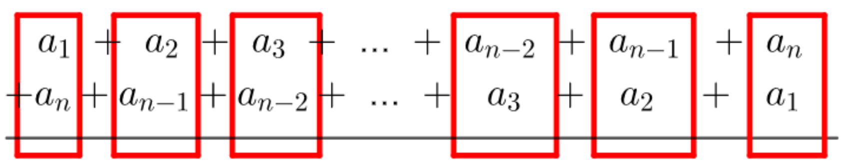
11.7 FINITE GEOMETRIC SERIES

LEARNING GOALS AND ESSENTIAL QUESTIONS

When you complete this investigation you should be able to answer the following questions and/or explain the following ideas.

- What is a geometric series?
- What is the technique for finding the sum of a geometric series? Why does that technique work? When generalized as a formula, what does each part of the formula represent?
- Why doesn't the technique work with other types of finite series?

In the previous investigation we learned a technique for summing finitely many terms in an **arithmetic series** by noticing a pattern that forms when the series is duplicated (doubling the sum) and terms are paired off so that the sums $a_1 + a_n$, $a_2 + a_{n-1}$, $a_3 + a_{n-2}$, … , $a_{n-1}+a_2$, and a_n+a_1 are all identical.



Since every pair has the same sum, $a_1 + a_n$ can represent the sum of ***any*** pair. There are n such pairs with a sum of $a_1 + a_n$ for a total sum of $n(a_1 + a_n)$ when all pairs are added together. But this represents twice the sum of the original series, so

$S_n = \frac{1}{2}n(a_1 + a_n)$ represents the sum of the original series.

This formula is very useful because we can add up very large series without having to write out and add up all of the many, many terms. It provides us with a great shortcut!

FINITE GEOMETRIC SERIES

In this investigation we will examine **GEOMETRIC SERIES** and determine techniques for summing finite geometric series.

A ***geometric series*** is the sum of the terms of a geometric sequence. For example, $1 + 5 + 25 + 125 + 625 + 3125$ is a geometric series since $1, 5, 25, 125, 625, 3125$ is a geometric sequence (the common ratio is 5).

A geometric series can have a finite number of terms or an infinite number of terms. If it has a finite number of terms, we call it a ***finite geometric series.***

The technique we developed for finite arithmetic series doesn't work for finite geometric series because their terms don't have a common difference. The common difference is what guaranteed that the pairs would always maintain the same sum (one term in each pair always increased by as much as the second term in the pair decreased from one pair to another).

A technique for finding the sum of a geometric series will rely instead on the common ratio of the term values. Let's see how this works.

SCALING A GEOMETRIC SERIES AND LOOKING FOR PATTERNS

Let's look at a small geometric series to search for patterns that could be useful: $2 + 6 + 18 + 54 + 162$.

The sequence's common ratio is $\frac{6}{2} = \frac{18}{6} = 3$

The sum of the series 242

- If we multiply all terms in the original series by 2: $2(2 + 6 + 18 + 54 + 162)$

  The new series becomes $4 + 12 + 36 + 108 + 324$

  With a sum of 484 which is 2 times larger than the original series.

- If we multiply all terms in the original series by 5: $5(2 + 6 + 18 + 54 + 162)$

  The new series becomes $10 + 30 + 90 + 270 + 810$

  With a sum of 1210 which is 5 times larger than the original series.

- If we multiply all terms in the original series by -4: $-4(2 + 6 + 18 + 54 + 162)$

  The new series becomes $-8 - 24 - 72 - 216 - 648$

  With a sum of -968 which is -4 times larger than the original series.

So far we have seen that multiplying a series by a constant (call it k) does two things.

- Every term in the series is k times as large

$$k[a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n] = ka_1 + ka_2 + ka_3 + \cdots + ka_{n-2} + ka_{n-1} + ka_n.$$

- The sum of the series is k times as large.

$$k[a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n] = k(sum\ of\ the\ original\ series)$$

We will see that these conclusions are key to developing a method for developing a shortcut to summing a finite geometric series.

SCALING A GEOMETRIC SERIES BY THE COMMON RATIO

Building from the previous section, what happens when the scalar we use is **ALSO** the common ratio? Let's continue to use the series $2 + 6 + 18 + 54 + 162$ and its common ratio 3.

- If we multiply all terms in the original series by 3: $3(2 + 6 + 18 + 54 + 162)$

  The new series becomes $6 + 18 + 54 + 162 + 486$

  With a sum of 726 which is 3 times larger than the original series.

Note that there were a lot of duplicated terms since each of the terms in the original series was created from multiplying another term in the series by the common ratio 3, and we just multiplied all of the terms in the series by the common ratio 3 to create the new series.

So how is this useful?

Let's align the two series by matching up the terms that are identical. Even though we know the sums (242 and 726 respectively), for right now we will use S_5 to represent the sum of the original series and $3 \cdot S_5$ to represent the sum of the new series.

$$\begin{aligned} S_5 &= 2 + 6 + 18 + 54 + 162 \\ 3 \cdot S_5 &= \phantom{2 + {}} 6 + 18 + 54 + 162 + 486 \end{aligned}$$

Now, what if we subtract the series (and their sums)?

$$\begin{aligned} S_5 &= 2 + 6 + 18 + 54 + 162 \\ -(3 \cdot S_5 &= \quad\; 6 + 18 + 54 + 162 + 486) \end{aligned}$$

Distribute -1 across all terms in the new series.

$$\begin{aligned} S_5 &= 2 + 6 + 18 + 54 + 162 \\ -3 \cdot S_5 &= \quad -6 - 18 - 54 - 162 - 486 \end{aligned}$$

Finally, we evaluate the differences between the term values and the series' sums. Notice that all of the duplicated terms "cancel" (their difference is

$$\begin{aligned} S_5 &= 2 + 6 + 18 + 54 + 162 \\ -3 \cdot S_5 &= \quad -6 - 18 - 54 - 162 - 486 \\ &\Downarrow \\ S_5 - 3 \cdot S_5 &= 2 + 0 + 0 + 0 + 0 - 486 \\ &\Downarrow \\ S_5 - 3 \cdot S_5 &= 2 - 486 \end{aligned}$$

So $S_5 - 3 \cdot S_5$ is the difference between the two series' sums, which must have the same value as the difference in the term values, or $2 - 486 = -484$.

$$\underbrace{S_5 - 3 \cdot S_5}_{\text{difference in series' sums}} = \overbrace{\underbrace{2 - 486}_{\text{difference in term values}}}^{\text{difference in series' sums}}$$

$$S_5 - 3 \cdot S_5 = 2 - 486$$

$$S_5 - \mathbf{3} \cdot S_5 = 2 - \mathbf{3}(162)$$

Again, even though we know the value of S_5 here (it's 242), let's pretend we don't just for the sake of demonstrating something important.

$$\begin{aligned} S_5 - 3 \cdot S_5 &= 2 - 486 \\ S_5 - 3 \cdot S_5 &= -484 \\ -2 \cdot S_5 &= -484 && \text{combine like terms on the left} \\ S_5 &= 242 && \text{divide by } -2 \end{aligned}$$

$S_5 = 242$, which we already knew, but this entire process demonstrates the technique for finding the sum of a finite geometric sequence without actually adding up all of the terms.

The following video is a review this process and the related ideas.

https://www.rationalreasoning.net/video/direct/view.php?s=20&/geometricseriesintro.mp4

FINDING THE SUM OF FINITE GEOMETRIC SERIES WITH A POSITIVE COMMON RATIO

Use the technique developed above to find the sum of the geometric series $7 + 21 + 63 + \cdots + 45927 + 137781$.

- The common ratio is $\frac{21}{7} = 3$

$$S_n - \mathbf{3} \cdot S_n = 7 - \mathbf{3}(137781)$$

$$-\mathbf{2} \cdot S_n = 7 - \mathbf{3}(137781)$$

$$S_n = \frac{7 - \mathbf{3}(137781)}{-2}$$

$$S_n = 206668$$

To find the sum of a finite geometric series defined by a given formulas, it might be helpful to write out the first few and the last few terms in each series.

Find the sum of the first 7 terms of the sequence defined by $a_n = 2(6)^{n-1}$

- The common ratio is 6
- The first term is $a_1 = 2$
- The 7th term is $a_7 = 2(6)^{7-1} = 2(6)^6 = 93312$

$$S_7 - \mathbf{6} \cdot S_7 = 2 - \mathbf{6}(93312)$$

$$-\mathbf{5} \cdot S_n = 2 - \mathbf{6}(93312)$$

$$S_n = \frac{2 - \mathbf{6}(93312)}{-5}$$

$$S_n = 111974$$

FINDING THE SUM OF FINITE GEOMETRIC SERIES WITH A NEGATIVE COMMON RATIO

The technique also works when the common ratio is negative (which causes the terms in the series to alternate between positive and negative values). Consider the series $6 - 12 + 24 - 48 + 96 - 192 + 384 - 768$ with a common ratio of $\frac{-12}{6} = \frac{24}{-12} = -2$. We form a new series by scaling the original series (and its sum) by -2

$$\begin{aligned} -2 \cdot S_8 &= -2(6 - 12 + 24 - 48 + 96 - 192 + 384 - 768) \\ &= -12 + 24 - 48 + 96 - 192 + 384 - 768 + 1536 \end{aligned}$$

We then set up the difference between the two series' sums and term values.

$$\begin{aligned} S_8 &= 6 - 12 + 24 - 48 + 96 - 192 + 384 - 768 \\ -(-2 \cdot S_8 &= \quad -12 + 24 - 48 + 96 - 192 + 384 - 768 + 1536) \end{aligned}$$

Distribute -1 across all the terms in the new series.

$$\begin{aligned} S_8 &= 6 - 12 + 24 - 48 + 96 - 192 + 384 - 768 \\ +2 \cdot S_8 &= \quad +12 - 24 + 48 - 96 + 192 - 384 + 768 - 1536 \end{aligned}$$

When we subtract the two series (and their sums), what is left? Fill in the missing parts in the following equation, and then fill in the missing parts in the steps to solve for S_n. Tip: The sum of this series should be a negative number!

$$S_8 - -2 \cdot S_8 = 6 - 1536$$

$$S_8 + 2 \cdot S_8 = -1530$$

$$3 \cdot S_8 = -1530$$

$$S_8 = \frac{-1530}{3}$$

It might also be helpful to write out the first few and the last few terms in each series if they aren't given. Be very careful with tracking signs in your calculations.

Find the sum of the first 11 terms of the sequence defined by $a_n = -4096\left(-\frac{3}{2}\right)^{n-1}$

- The common ratio is -32
- The first term is $a_1 = -4096$
- The 11th term is $a_{11} = -4096\left(-\frac{3}{2}\right)^{11-1} = -236196$

$$S_{11} - -\mathbf{\frac{3}{2}} \cdot S_{11} = -4096 - \mathbf{\frac{3}{2}}(-236196)$$

$$S_{11} + \mathbf{\frac{3}{2}} \cdot S_{11} = -4096 + 354294$$

$$\frac{5}{2} \cdot S_{11} = 350198$$

$$S_{11} = \frac{2}{5}(350198)$$

$$S_{11} = 140079.2$$

The following video discusses another example of finding the sums of finite geometric series using this technique, this time where the common ratio is a negative value.

https://www.rationalreasoning.net/video/direct/view.php?s=21&/geometricseriesnegative_r.mp4

GENERALIZING OUR REASONING TO CREATE A FORMULA

Let $a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n$ be a finite geometric series with a sum of S_n. If all of the terms of the series are multiplied by the common ratio r, then two things happen.

- Each of the terms becomes r times as large.
- The sum is r times as large.

In other words, we get the following:

$$r \cdot S_n = r \cdot a_1 + r \cdot a_2 + r \cdot a_3 + \cdots + r \cdot a_{n-2} + r \cdot a_{n-1} + r \cdot a_n$$

However, we also know that $r \cdot a_1 = a_2$ (because the common ratio is the value we multiply any term value by to produce the next term value), $r \cdot a_2 =$, $r \cdot a_3 = a_4$, and so on. So we could equivalently rewrite this as follows

$$r \cdot S_n = a_2 + a_3 + \cdots + a_n + r \cdot a_n$$

We then compare the original series (and its sum) to the new series (and its sum), aligning them so that the same term values match up.

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n \\ r \cdot S_n &= \qquad a_2 + a_3 + \cdots + \qquad\quad a_n + r \cdot a_n \end{aligned}$$

Taking the difference between the two series and sums results in

$$S_n - r \cdot S_n = a_1 - r \cdot a_n$$

We now solve for S_n

$$\begin{aligned} S_n - r \cdot S_n &= a_1 - r \cdot a_n \\ S_n(1 - r) &= a_1 - r \cdot a_n && \text{factor out } S_n \text{ from the left side} \\ S_n &= \frac{a_1 - r \cdot a_n}{1 - r} && \text{divide by } 1 - r \end{aligned}$$

The sum of a finite geometric series with n total terms (or the sum of the first n terms of an infinite geometric series) with a_1 as the first term value, a_n as the last term value, and r as the common ratio can be represented by the following formula.

$$S_n = \frac{a_1 - r \cdot a_n}{1 - r}$$

The formula works for any finite geometric series with any number of terms.

The following video repeats examples from earlier in the lesson using the formula and explains the connections between the formula and the technique we've practiced.

https://www.rationalreasoning.net/video/direct/view.php?s=1Z&/geometricseriesformula.mp4